Chapter 2: Functions of a Random Variable

2.1 Distributions of Functions of a Random Variable

**Theorem 1:** Let $X$ have cdf $F_X(x)$ and let $Y = g(X)$. Let $X$ and $Y$ be supports of $X$ and $Y$ respectively, i.e.
$$X = \{x: f_X(x) > 0\}$$
and
$$Y = \{y: y = g(x) > 0, \text{ for some } x \in X\}.$$ 

(a) If $g$ is an increasing function on $X$, then
$$F_Y(y) = F_X(g^{-1}(y)) \text{ for every } y \in Y.$$ 
(b) If $g$ is an decreasing function on $X$ and $X$ is a continuous r.v., then
$$F_Y(y) = 1 - F_X(g^{-1}(y)) \text{ for every } y \in Y.$$
Theorem 2: Let $X$ have pdf $f_X(x)$ and let $Y = g(X)$, where $g$ is a monotone function. Let $X$ and $Y$ be supports of $X$ and $Y$ respectively. Suppose that $f_X(x)$ is continuous on $X$ and that $g^{-1}(y)$ has a continuous derivative on $Y$. Then the pdf of $Y$ is given by

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|, & y \in \text{spt of } Y \\ 0, & \text{otherwise} \end{cases}$$

where "spt of $Y$" means the support of $Y$.

Theorem 3: (probability integral transformation)
Let $X$ have continuous cdf $F_X(x)$ and define r.v. $Y$ as $Y = F_X(X)$. Then $Y$ is uniformly distributed on $(0, 1)$, i.e.

$$P(Y \leq y) = y, \ 0 < y < 1.$$ 

2.2 Expectations

Definition 1: The expectation or mean of a r.v.
$g(X)$, denoted by $E(g(X))$, is defined as follows:

(a) If $X$ is discrete, then
\[ E(g(X)) = \sum_{x \in \text{spt of } X} g(x)f_X(x). \]

(b) If $X$ is continuous, then
\[ E(g(X)) = \int_{-\infty}^{\infty} g(x)f_X(x) \, dx. \]

**Theorem 4:** Let $X$ be a r.v. and let $a, b$ and $c$ be constants, then for any functions $g_1(x)$ and $g_2(x)$ such that $E(g_1(X))$ and $E(g_2(X))$ exist, the following properties are satisfied:

(a)
\[ E(ag_1(X) + bg_2(X) + c) = aE(g_1(X)) + bE(g_2(X)) + c. \]

(b) If $g_1(x) \geq 0$ for all $x$, then $E(g_1(X)) \geq 0$.

(c) If $g_1(x) \geq g_2(x)$ for all $x$, then
\[ E(g_1(X)) \geq E(g_2(X)). \]

(d) If $a \leq g_1(x) \leq b$ for all $x$, then
\[ a \leq E(g_1(X)) \leq b. \]
2.3 Moments and Moment Generating Functions

Definition 2: For each integer \( n \), the \( n \)th moment of \( X \), denoted by \( \mu'_n \), is

\[
\mu'_n = E(X^n).
\]

The \( n \)th central moment of \( X \), denoted by \( \mu_n \), is

\[
\mu_n = E(X - \mu)^n
\]

where \( \mu = \mu'_1 = E(X) \).

Definition 3: The second central moment of a r.v. \( X \) is called the variance of \( X \), i.e.

\[
Var(X) = E(X - E(X))^2
\]

Theorem 5: If \( X \) is a r.v. with finite variance, then

(a) \[
Var(X) = E(X^2) - (E(X))^2
\]

(b) For any constants \( a \), and \( b \),
\[
Var(aX + b) = a^2 Var(X)
\]

**Definition 4:** Let \( X \) be a r.v. with cdf \( F_X(x) \). The **moment generating function (mgf)** of \( X \) denoted by \( M_X(t) \), is
\[
M_X(t) = E(e^{tx}),
\]
provided that the expectation exists for \( t \) in some neighborhood of 0; otherwise, we said that the moment generating function does not exist.

**Theorem 6:** Let \( a \), and \( b \) be any constants. Then the mgf of a r.v. \( aX + b \) is given by
\[
M_{aX+b}(t) = e^{bt} M_X(at).
\]

**Theorem 7:** If \( X \) has mgf \( M_X(t) \), then
\[
E(X^n) = M_X^{(n)}(0) \equiv \left. \frac{d^n}{dt^n} M_X(t) \right|_{t=0}
\]

**Theorem 8:** Let \( F_X(y) \) and \( F_Y(y) \) be two cdfs all
of whose moments exist.

(a) If $X$ and $Y$ have bounded support, then

$$F_X(u) = F_Y(u) \text{ for all } u,$$

if and only if

$$E(X^r) = E(Y^r) \text{ for all } r = 0, 1, 2, \ldots.$$

(b) If the mgfs exist and $M_X(t) = M_Y(t)$ for all $t$ in some neighborhood of 0, then $F_X(u) = F_Y(u)$ for all $u$.

**Theorem 9: (Convergence of mgfs)** Let $X_1, X_2, \ldots$ be a sequence of r.v.s, each with mgf $M_{X_i}(t)$.

Suppose that there exist an mgf $M_X(t)$ such that

$$\lim_{i \to \infty} M_{X_i}(t) = M_X(t) \text{ for all } t \text{ in a neighborhood of } 0,$$

then there exist a unique cdf $F_X$ whose moments are determined by $M_X(t)$ and for all $x$ where $F_X(x)$ is continuous, we have

$$\lim_{i \to \infty} F_{X_i}(x) = F_X(x).$$

That is, convergence of mgfs to an mgf in a neighborhood of 0, implies convergence of cdfs.